Lecturered By Dr. H.K. Yadav.
Deptt. of Mathematics.
Marwari College, Darbhanga.
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Differential Calculus (partial differentiation)

Q.1. If $u = \cos^{-1}\frac{x-y}{x+y}$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Soln: We have $\cos u = \frac{x-y}{x+y}$

It is clear that $\frac{x-y}{x+y}$ is of zero degree and hence $\cos u$ is a homogeneous function of ~~degree~~ zero degree.

$\therefore$ By Euler's theorem,

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{(\cos u)}{\partial y} = n\cos u = 0\times\cos u = 0$$

$$\Rightarrow -x\sin u.\frac{\partial u}{\partial x} - y\sin u.\frac{\partial u}{\partial y} = 0 \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Proved

Q.2. If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Soln: We have $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Since, $\sin u$ is a homogeneous function of zero degree.

$\therefore$ By Euler's theorem,

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial}{\partial y}(\sin u) = 0\times\sin u$$

$$\Rightarrow \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Proved

Q.3. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Soln: We have, $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos u = \frac{x(1+\frac{y}{x})}{\sqrt{x}(1+\sqrt{\frac{y}{x}})}$$

$$\Rightarrow \cos u = \frac{x^{\frac{1}{2}}(1+\frac{y}{x})}{1+\sqrt{\frac{y}{x}}} = v \text{ (say)} \quad \text{—— I}$$

We find that v is a homogeneous function of two independent variable x and y and of degree $\frac{1}{2}$. Hence applying Euler's theorem for the function v we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = \frac{1}{2}v \quad \text{—— II}$$

From I, $v = \cos u$

$$\therefore \frac{\partial v}{\partial x} = -\sin u\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y} = -\sin u\frac{\partial u}{\partial y}$$

Substituting these values in II, we get

$$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Proved

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Differential calculus (partial differentiation)

Q.4. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Soln: $\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = v$ (say)

Since v is a homogeneous function of x and y of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$ proved.

Q.5. If $u = \sin^{-1}(\sqrt{x}+\sqrt{y})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(\sqrt{x}+\sqrt{y})\cot(\sqrt{x}+\sqrt{y})$$

Soln: We have, $u = \sin(\sqrt{x}+\sqrt{y}) \Rightarrow \sin^{-1}u = \sqrt{x}+\sqrt{y}$

$$\Rightarrow \sin^{-1}u = \sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right) = V \text{ (say)} \quad \text{—— (I)}$$

We find that V is a homogeneous function of two independent variables x and y of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nV = \frac{1}{2}V \quad \text{—— (II)}$$

From (I), $V = \sin^{-1}u$

$$\therefore \frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y}$$

Substituting these two values in II, we get

$$\frac{x}{\sqrt{1-u^2}}\frac{\partial u}{\partial x} + \frac{y}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} = \frac{1}{2}\sin^{-1}u$$

$$\Rightarrow \frac{1}{\sqrt{1-u^2}}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}(\sqrt{x}+\sqrt{y})$$

Rest part of Q.5.

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}(\sqrt{x}+\sqrt{y})\sqrt{1-u^2}$$

$$=\frac{1}{2}(\sqrt{x}+\sqrt{y})\sqrt{1-\sin^2(\sqrt{x}+\sqrt{y})}$$

Hence, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}(\sqrt{x}+\sqrt{y})\cos(\sqrt{x}+\sqrt{y})$ Proved

Q.6. If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

Soln: We have, $u=\sin^{-1}\frac{x^2+y^2}{x+y}$

$$\Rightarrow \sin u=\frac{x^2+y^2}{x+y}=\frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}=x\phi\left[\frac{y}{x}\right]=v,\text{ say}$$

Now, v is a homogeneous function of degree 1

∴ By Euler's theorem, we get

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1.v$$

But $v=\sin u$

$$\therefore \frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}$$

Substituting these values in (I), we get

$$\therefore x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$ Proved.

Q.7. If $\sin u=\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$

Soln: Since, $\sin u=\frac{x^3+y^3}{x-y}$

Here, we see that $\sin u$ is a homogeneous function of the second degree.

∴ By Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=2\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=2\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$$ Proved